RIEMANN-HILBERT METHODS IN THE THEORY OF ORTHOGONAL POLYNOMIALS

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To Barry Simon, on his 60th birthday.
Mathematician extraordinaire, teacher and friend.

Abstract. In this paper we describe various applications of the Riemann-Hilbert method to the theory of orthogonal polynomials on the line and on the circle.

1. Introduction

In this paper $d\mu$ denotes either a Borel measure on $\mathbb{R}$ with finite moments

$$\int_{\mathbb{R}} |x|^m \, d\mu(x) < \infty \ , \qquad m = 0, 1, 2, \dots \tag{1}$$

or a finite Borel measure on the unit circle S^1

$$\int_{S^1} d\mu(\theta) < \infty. \tag{2}$$

In addition, unless stated explicitly otherwise, we will always assume that $d\mu$ is a *nontrivial probability measure*, i.e. $\text{supp}(d\mu)$ is infinite and the integral of $d\mu$ is 1.

Let

$$p_n(x) = k_n x^n + \dots \ , \qquad k_n > 0 \ , \qquad n = 0, 1, 2, \dots \tag{3}$$

$$\phi_n(z) = \kappa_n z^n + \dots \ , \qquad \kappa_n > 0 \ , \qquad n = 0, 1, 2, \dots \tag{4}$$

denote the orthonormal polynomials (OP's) with respect to $d\mu$ on $\mathbb{R}$ and S^1 respectively (see [Sze]),

$$\int_{\mathbb{R}} p_n(x) p_m(x) d\mu(x) = \int_{S^1} \overline{\phi_n(e^{i\theta})} \, \phi_m(e^{i\theta}) d\mu(\theta) = \delta_{n,m} \ , \qquad n, m \geq 0. \tag{5}$$

The fact that $d\mu$ is nontrivial implies, in particular, that the p_n's, and the ϕ_n's, exist and are unique for all $n \geq 0$.

As is well known, the p_n's satisfy a three-term recurrence relation

$$b_{n-1} p_{n-1}(x) + a_n p_n(x) + b_n p_{n+1}(x) = x p_n(x) \ , \qquad n \geq 0 \tag{6}$$

where

$$a_n \in \mathbb{R} \ , \qquad b_n > 0 \ , \qquad n \geq 0 \tag{7}$$

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and $b_{-1} \equiv 0$. Similarly the ϕ_n's satisfy the Szegö recurrence relation

$$\sqrt{1-|\alpha_n|^2}\ \phi_{n+1}(z) = z\phi_n(z) - \bar{\alpha}_n \phi_n^*(z)\ , \qquad n \geq 0 \tag{8}$$

where

$$\alpha_n \in \mathbb{C}\ , \qquad |\alpha_n| < 1\ , \qquad n \geq 0 \tag{9}$$

and for any polynomial $q(z)$ of degree n

$$q^*(z) \equiv z^n\ \overline{q(1/\bar{z})} \tag{10}$$

denotes the so-called *reverse polynomial*. Following [Sim2], we call the α_n's *Verblunsky coefficients*. A simple computation shows that

$$\alpha_n = -\frac{1}{\kappa_{n+1}} \overline{\phi_{n+1}(0)}\ , \qquad n \geq 0\ . \tag{11}$$

On $\mathbb{R}$ we define the $(n+1)\times(n+1)$ *Hankel determinant*

$$D_n = \det\left(\int_{\mathbb{R}} x^{j+k} d\mu(x)\right)_{0\leq j,k\leq n}\ , \qquad n \geq 0\ , \tag{12}$$

and on S^1 we similarly define the $(n+1)\times(n+1)$ *Toeplitz determinant*

$$\Delta_n = \det\left(\int_{S^1} e^{-i(j-k)\theta}\, d\mu(\theta)\right)_{0\leq j,k\leq n}\ , \qquad n \geq 0. \tag{13}$$

The determinants D_n and Δ_n are closely related to the OP's $\{p_n\}, \{\phi_n\}$ respectively: Indeed one has (see for example [Sze])

$$\frac{D_{n-1}}{D_n} = k_n^2\ , \qquad \frac{\Delta_{n-1}}{\Delta_n} = \kappa_n^2\ , \qquad n \geq 1\ . \tag{14}$$

Given $d\mu$, the study of the algebraic and asymptotic properties of the quantities

$$a_n, b_n, p_n(x), k_n, \alpha_n, \phi_n(z), \kappa_n\ ,$$

and also

$$D_n \quad \text{and} \quad \Delta_n\ ,$$

constitutes the core of the classical theory of orthogonal polynomials.

The three-term relation (6) can be re-written in the form

$$Lp(z) = zp(z)\ , \qquad p(z) = (p_0(z), p_1(z), p_2(z), \ldots)^T\ , \tag{15}$$

where L is an infinite Jacobi matrix, i.e. L is symmetric and tridiagonal

$$L = \begin{pmatrix} a_0 & b_0 & & \\ b_0 & a_1 & b_1 & 0 \\ & b_1 & a_2 & \ddots \\ 0 & & \ddots & \ddots \end{pmatrix} \tag{16}$$

with $b_i > 0$, $i \geq 0$. In the case that $d\mu$ has compact support on $\mathbb{R}$, the operator L is bounded on

$$\ell_2^+ = \left\{ u = (u_0, u_1, \ldots)^T :\ \sum_{i=0}^{\infty} |u_i|^2 < \infty \right\}.$$

Let

$$F : \{d\mu \text{ on } \mathbb{R} :\ \operatorname{supp}(d\mu) \text{ compact}\} \to \{\text{bounded Jacobi matrices on } \ell_2^+\}.$$

denote the map taking $d\mu \mapsto L$. Conversely, if L is a bounded Jacobi matrix, then in particular L is self-adjoint, and we let $d\mu$ denote the spectral measure associated with L in the cyclic subspace generated by L and e_0, where $e_0 = (1, 0, 0, \ldots)^T \in \ell_2^+$. Thus

$$\left(e_0, \frac{1}{L-\lambda} e_0\right) = \int \frac{d\mu(x)}{x-\lambda}\ , \qquad \lambda \in \mathbb{C}\backslash\mathbb{R} \tag{17}$$

and it follows further that $d\mu$ has compact support. Let

$$\hat{F}:\ \{\text{bounded Jacobi matrices on } \ell_2^+\} \to \{d\mu \text{ on } \mathbb{R}:\ \operatorname{supp}(d\mu) \text{ compact}\}$$

denote the map taking L to $d\mu$. The basic fact of the matter (see, for example, [A], [Sim1], and also [D2]) is that F and $\hat{F}$ are inverse to each other, $F \circ \hat{F} = id$, $\hat{F} \circ F = id$. From this point of view the (classical) orthogonal polynomial problem is the inverse spectral component of a spectral/inverse spectral problem. If the support of $d\mu$ is not compact, then the situation is similar, but the relation between $d\mu$ and L is more complicated because L is now an unbounded operator and we must distinguish between different self-adjoint extensions of L (see [A], [Sim1] for more details).

In the case of measures $d\mu$ on the unit circle, the role of the Jacobi matrices is played by so-called CMV matrices C (see [Sim2]). Such matrices C are unitary in ℓ_2^+ and pentadiagonal, and have the form

$$C = LM \tag{18}$$

where L and M are block diagonal

$$L = \operatorname{diag}(\Theta_0, \Theta_2, \Theta_4, \ldots\,)\ , \qquad M = \operatorname{diag}(1, \Theta_1, \Theta_3, \ldots\,) \tag{19}$$

with

$$\Theta_j = \begin{pmatrix} \bar{\alpha}_j & \rho_j \\ \rho_j & -\alpha_j \end{pmatrix}\ , \qquad j \geq 0\ . \tag{20}$$

Here

$$|\alpha_j| < 1\ , \qquad j \geq 0 \tag{21}$$

and

$$\rho_j = \sqrt{1 - |\alpha_j|^2}\,. \tag{22}$$

CMV matrices are named for Cantero, Moral and Velázquez [CMV], but in fact they appeared earlier in the literature (see, in particular, [Wat]). Let

$$\psi:\ \{d\mu \text{ on } S^1\} \to \{\text{CMV matrices}\}$$

denote the map taking $d\mu \to C$, the CMV matrix constructed from the Verblunsky coefficients $\alpha_j = \alpha_j(d\mu)$, $j \geq 0$, of $d\mu$, according to (18), (19) and (20). Conversely, given a CMV matrix C, let $d\mu$ be the spectral measure associated with C in the cyclic subspace generated by C, $C^* = C^{-1}$ and e_0. Let

$$\hat{\psi}:\ \{\text{CMV matrices}\} \to \{d\mu \text{ on } S^1\}$$

denote the map taking C to $d\mu$. Then, as above (see [Sim2]), ψ and $\hat{\psi}$ are inverse to each other, and we see again that the classical orthogonal polynomial problem on S^1 is the inverse spectral component of a spectral/inverse spectral problem.

The techniques used to analyze the direct spectral maps, $\hat{F}$ and $\hat{\psi}$, are generally very different from the techniques used to analyze the inverse spectral maps, F or ψ, though sometimes there is some overlap (see e.g. [DK]). It is also interesting to note that in the solution of integrable systems one needs knowledge of *both* $\hat{F}$ and

F (or $\hat{\psi}$ and ψ). For example, the Toda lattice induces a flow $L_0 \mapsto L = L(t)$ on Jacobi matrices ([F])

$$\begin{aligned} \frac{dL}{dt} &= B(L)L - LB(L) \\ L(t=0) &= L_0 \end{aligned} \tag{23}$$

where

$$L = \begin{pmatrix} a_0 & b_0 & & 0 \\ b_0 & a_1 & b_1 & \\ & b_1 & a_2 & \ddots \\ 0 & & \ddots & \ddots \end{pmatrix}, \quad B(L) = \begin{pmatrix} 0 & b_0 & & 0 \\ -b_0 & 0 & b_1 & \\ & -b_1 & 0 & \ddots \\ 0 & & \ddots & \ddots \end{pmatrix},$$

and the solution of (23) is given by the following well-known procedure ([M]):

$$L_0 \xrightarrow{\hat{F}} d\mu_0 = \hat{F}(L_0) \to d\mu_t(\lambda) = \frac{e^{2\lambda t} d\mu_0(\lambda)}{\int_{\mathbb{R}} e^{2xt} d\mu_0(x)} \xrightarrow{F} L(t) = F(d\mu_t)$$

The analysis of $\hat{F}$ and $\hat{\psi}$ has benefited greatly from the powerful developments that have taken place over many years in the spectral theory of Schrödinger operators and their discrete analogs, reaching, over the last 20 years or so, and in the case of one dimension, a state of great precision. Here Barry Simon and his school have played a decisive role, and we refer the reader to [Sim2], in particular, Part 2. The systematic analysis of F begins with the classic memoir of Stieltjes 1894-1895. Up till that point, a great deal of information had been obtained concerning particular polynomials, such as Legendre polynomials, Jacobi polynomials, Hermite polynomials, etc., but a unified point of view based on the orthogonality relation (5) had not yet emerged. The analysis of ψ began in 1920, when Szegö initiated the systematic study of polynomials orthogonal with respect to a measure on S^1, as in (5). Szegö's work in turn has led to many remarkable developments by researchers from all over the world, particularly the former USSR, Europe and the USA. We refer the reader to Simon's book [Sim2], where these developments are discussed in great detail together with many fascinating anecdotes concerning their discovery. Starting in the early 1950's with the celebrated work of Gel'fand and Levitan, various techniques were developed to recover one-dimensional Schrödinger operators from their spectral measures. In the 1970's, techniques based on the inverse-Schrödinger method (see [CGe] and [F]) started to play a role in the analysis of F and ψ. The goal of this paper is to describe one of these techniques, which is different from the techniques in [CGe] or [F], and which has proved extremely fruitful, *viz.*, the Riemann-Hilbert (RH) method, also referred to as the Riemann-Hilbert Problem (RHP). The scope of the paper is limited to describing results for F and ψ obtained by RHP. Some of the results that we describe are quite standard and are included only for purposes of illustration. Other results, particularly asymptotic results, have been obtained, so far, only through RH methods. For a full up-to-date discussion of what is known about F and ψ, including the seminal contributions of Golinskii, Ismail, Khrushchev, Lubinsky, Nevai, Rakhmanov, Saff, Totik and many others, we again refer the reader to [Sim1] and [Sim2].

To begin, let Σ be an oriented contour in the complex plane $\mathbb{C}$ (see Figure 1). By convention, if we move along the contour in the direction of the orientation, the $(+)$-side (resp. $(-)$-side) of the contour lies to the left (resp. right) (see again

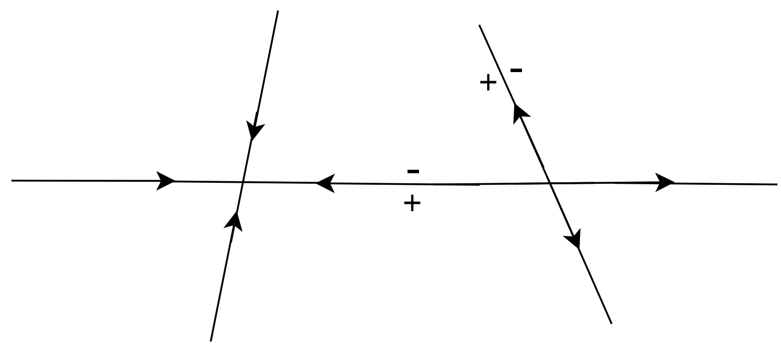

FIGURE 1. The contour Σ

Figure 1). A $k \times k$ *jump matrix* v on Σ is a mapping from $\Sigma \to G\ell(k, \mathbb{C})$ such that $v, v^{-1} \in L^\infty(\Sigma)$. We say that an $\ell \times k$-valued matrix function $m(z)$ is a solution of the RHP (Σ, v) if

(a) $m(z)$ is analytic in $\mathbb{C} \backslash \Sigma$
(b) $m_+(z) = m_-(z)v(z)$, $z \in \Sigma$, where $m_\pm(z) = \lim_{z' \to z, z \in (\pm)-\text{side}} m(z')$

If in addition $\ell = k$ and

(c) $m(z) \to I$ as $z \to \infty$

we say that m is a solution of the *normalized* RHP (Σ, v).

Many technical issues arise. For example, in what sense do the limits $m_\pm$ exist? In what sense does $m(z) \to I$ in (c)? How should one understand (b) at points of self-intersection in Σ? Under what assumptions on Σ and v does a solution $m(z)$ exist? And if we normalize as in (c), is the solution unique? We will not consider such issues here and in the text that follows, and we simply refer the reader to [CG] and the references therein for a general discussion of RHP's (see also [DZ4] for more recent information, and [BDT] for a discussion of points of self-intersections). In this paper we will consider almost exclusively problems with solutions $m(z)$ that are analytic in $\mathbb{C} \backslash \Sigma$ and continuous up to the boundary and at $z = \infty$. For such solutions, the limits in (b) and (c) are taken pointwise. Furthermore, for the problems we consider, the solution of the normalized RHP will always exist and be unique.

At the analytical level, a normalized RHP is equivalent to a problem for coupled singular integral equations on Σ. This is seen as follows.

Let C^Σ denote the Cauchy operator on Σ,

$$C^\Sigma h(z) \equiv \int_\Sigma \frac{h(s)}{s - z} \frac{ds}{2\pi i} , \qquad z \in \mathbb{C} \backslash \Sigma \tag{24}$$

with boundary values

$$(C^\Sigma_\pm h)(z) = \lim_{\substack{z' \to z \\ z' \in (\pm)-\text{side}}} (C^\Sigma h)(z') , \qquad z \in \Sigma . \tag{25}$$

Under reasonable conditions on Σ, $C^\Sigma_\pm \in \mathcal{L}(L^p(\Sigma))$, the bounded operators from $L^p(\Sigma) \to L^p(\Sigma)$, for any $1 < p < \infty$, and we have the relation

$$C^\Sigma_+ - C^\Sigma_- = 1 . \tag{26}$$

Let

$$v(z) = (v_-(z))^{-1} v_+(z)\ , \qquad z \in \Sigma \tag{27}$$

be any pointwise factorization of v where

$$v_\pm,\ (v_\pm)^{-1} \in L^\infty(\Sigma)\ . \tag{28}$$

Set

$$\begin{cases} w_+ = v_+ - I, \quad w_- = I - v_- \\ \qquad w = (w_+, w_-) \end{cases} \tag{29}$$

and define the singular integral operator on Σ

$$C_w^\Sigma h \equiv C_+^\Sigma(hw_-) + C_-^\Sigma(hw_+) \tag{30}$$

for row k-vectors h. As $w_\pm \in L^\infty(\Sigma)$, $C_w^\Sigma \in \mathcal{L}(L^p(\Sigma))$, $1 < p < \infty$. Suppose in addition that

$$w_\pm \in L^p(\Sigma) \ \text{ for some } \quad 1 < p < \infty\ , \tag{31}$$

and consider the equation for a $k \times k$-matrix function μ

$$(1 - C_w^\Sigma)\mu = I \tag{32}$$

in $I + L^p(\Sigma)$, or more precisely

$$(1 - C_w^\Sigma)\nu = C_w^\Sigma I = C_+^\Sigma w_- + C_-^\Sigma w_+ \in L^p(\Sigma) \tag{33}$$

where

$$\mu = I + \nu\ , \qquad \nu \in L^p. \tag{34}$$

If a solution $\mu = I + \nu$ of (32)–(34) exists, set

$$m(z) = I + C^\Sigma(\mu(w_+ + w_-))(z). \tag{35}$$

Then a simple calculation shows that $m_\pm = \mu v_\pm$, and hence $m_+ = m_- v$, and as $m(z) \to I$ as $z \to \infty$, we see that (35) gives a solution of the normalized RHP (Σ, v). Thus the normalized RHP (Σ, v) reduces to the analysis of the singular integral equations (32).

The connection between the OP problem and the RHP is due to Fokas, Its and Kitaev [FIK]. Let

$$P_n = \frac{1}{k_n} p_n = x^n + \dots\ , \qquad n \geq 0 \tag{36}$$

denote the monic orthogonal polynomials associated with a measure

$$d\mu(x) = w(x)dx\ , \qquad w(x) \geq 0 \tag{37}$$

absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$, with $x^j w(x) \in H^1(\mathbb{R})$, the first Sobolev space, for all $j \geq 0$. Let $\Sigma = \mathbb{R}$, oriented from $-\infty$ to $+\infty$, and equipped with jump matrix

$$v = v(x) = \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}\ , \qquad -\infty < x < \infty. \tag{38}$$

Finally, for any $n \geq 0$, let $X^{(n)} = (X_{ij}^{(n)})_{1 \leq i,j \leq 2}$ solve the RHP $(\mathbb{R}, v)$

$$X^{(n)}(z) \ \text{ analytic in } \ \mathbb{C}\backslash\mathbb{R}$$

$$X_+^{(n)}(z) = X_-^{(n)}(z) v(z)\ , \qquad z \in \mathbb{R} \tag{39}$$

normalized so that

$$X^{(n)}(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} \to I \ \text{ as } \ z \to \infty\ .$$

Then ([FIK], in addition see [D2]) direct computation shows that

$$X^{(n)}(z) = \begin{pmatrix} P_n(z) & C(P_n w)(z) \\ -2\pi i\ k_{n-1}^2 P_{n-1}(z) & -2\pi i\ k_{n-1}^2 C(P_{n-1}w)(z) \end{pmatrix} \tag{40}$$

where $C = C^{\mathbb{R}}$ denotes the Cauchy operator on $\Sigma = \mathbf{R}$. In particular,

$$P_n(z) = X_{11}^{(n)}(z)\ . \tag{41}$$

Furthermore, if $X_1^{(n)}$ denotes the residue of $X^{(n)} \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix}$ at infinity,

$$X^{(n)}(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} = I + \frac{X_1^{(n)}}{z} + O\left(\frac{1}{z^2}\right),$$

then

$$k_{n-1}^2 = -\frac{1}{2\pi i}(X_1^{(n)})_{21} \tag{42}$$

and in the notation of (6)

$$a_n = (X_1^{(n)})_{11} - (X_1^{(n+1)})_{11} \tag{43}$$

$$b_{n-1}^2 = (X_1^{(n)})_{12}(X_1^{(n+1)})_{21} \tag{44}$$

Also by (14) and (42),

$$\frac{D_{n-1}}{D_n} = -\frac{1}{2\pi i}(X_1^{(n+1)})_{21} \tag{45}$$

Thus all the basic quantities of interest in the OP problem can be read off from the solution $X^{(n)}$ of the RHP $(\mathbb{R}, v)$ above.

On the unit circle, the situation is similar. Let

$$\Phi_n = \frac{1}{\kappa_n}\phi_n = z^n + \dots\ , \qquad n \geq 0 \tag{46}$$

denote the monic orthogonal polynomials associated with a measure

$$d\mu(\theta) = \omega(\theta)\frac{d\theta}{2\pi} \tag{47}$$

absolutely continuous with respect to Lebesgue measure on S^1 with $\omega(\theta) \in H^1(S^1)$, $\omega(\theta) = \omega(\theta + 2\pi)$. Fix $n \geq 0$ and let $\Sigma = S^1$, oriented counterclockwise. Equip S^1 with the jump matrix

$$v = v(\theta) = \begin{pmatrix} 1 & \omega(\theta)z^{-n} \\ 0 & 1 \end{pmatrix}\ , \qquad z = e^{i\theta} \tag{48}$$

and let $Y^{(n)} = (Y_{ij}^{(n)})_{1\leq i,j\leq 2}$ solve the RHP (S^1, v)

- $Y^{(n)}(z)$ analytic in $\mathbb{C} \setminus S^1$ (49)
- $Y_+^{(n)}(z) = Y_-^{(n)}(z)v(\theta)\ , \quad z = e^{i\theta} \in S^1$ (50)

normalized so that

- $Y^{(n)}(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} \to I$ as $z \to \infty$. (51)

Then again (cf. [BDJ]) direct computation shows that

$$Y^{(n)}(z) = \begin{pmatrix} \Phi_n(z) & C(\Phi_n\ \omega/S^n)(z) \\ -\kappa_{n-1}^2 \Phi_{n-1}^*(z) & -\kappa_{n-1}^2 C(\Phi_{n-1}^*\ \omega/S^n)(z) \end{pmatrix} \tag{52}$$

where $C = C^{S^1}$ denotes the Cauchy operator on S^1 and Φ_{n-1}^* is the reverse polynomial as in (10). In particular

$$\Phi_n(z) = Y_{11}^{(n)}(z) \tag{53}$$

and hence by (11),

$$\alpha_{n-1} = -\overline{Y_{11}^{(n)}(z=0)}\ . \tag{54}$$

Also

$$\kappa_{n-1}^2 = -Y_{21}^{(n)}(z=0) \tag{55}$$

and hence

$$\frac{\Delta_{n-2}}{\Delta_{n-1}} = -Y_{21}^{(n)}(z=0). \tag{56}$$

Again we see that all basic quantities in the OP problem on the circle are expressed in terms of the solution $Y^{(n)}$ of the RHP (S^1, v).

The outline of the paper is as follows. In Section 2 we show how to use the RHP's $(\mathbb{R}, v)$ and (S^1, v) above to derive various identities, equations and formulae for the OP problem. In Section 3 we describe the application of the steepest descent method of Deift-Zhou for RHP's to asymptotic problems for OP's. Finally, in Section 4 we describe the application of RH ideas to areas related to the OP problem, such as random matrix theory, multi-orthogonal polynomials, orthogonal Laurent polynomials, and the rarefaction problem for the Toda lattice.

Technical Remark In most of the paper we will be considering probability measures with some degree of smoothness as in (37) and (47) above. For such weights we then use the RHP's to derive, in particular, various identities such as (6), (8), (85), etc. If $d\mu(x)$ is an arbitrary probability measure on $\mathbb{R}$ with finite moments, or $d\mu(\theta)$ is a probability measure on S^1, we can approximate $d\mu(x)$ and $d\mu(\theta)$ appropriately with smooth measures $d\mu_\epsilon(x)$ and $d\mu_\epsilon(\theta)$ respectively: For such measures (6), (8), (85), etc., are true, and letting $\epsilon \downarrow 0$ we conclude that these identities are true, as they should be, for all measures $d\mu(x)$ and $d\mu(\theta)$ as above. Similar considerations apply at many points in the paper and we leave the details to the interested reader.

2. Applications of $(\mathbb{R}, v)$ and (S^1, v): identities, equations and formulae

The applications of Riemann-Hilbert techniques to OP's are principally of two types:

(a) algebraic
(b) asymptotic.

Under (a), the goal is to derive identities, equations and useful formulae for the OP problems. Under (b), the goal is to determine the asymptotic behavior of the OP's p_n, P_n, ϕ_n, Φ_n as $n \to \infty$: Here one considers the case where the weight $\omega(x)$ is independent of n, as well as the case where $\omega(x)$ depends on n in a prescribed fashion (see (106) below). We consider (a) in this section, and (b) in the next.

Regarding (a), there is a general methodology, which may be traced all the way back to the original work of Gel'fand and Levitan, and which may be stated loosely as follows: If the jump matrix for a RHP is independent of a parameter, then differentiation with respect to that parameter (or taking differences in the discrete case) leads to an equation/identity.

We illustrate this methodology, which may be viewed as the analog for RHP's of the celebrated theorem of Noether on conserved quantities for dynamical systems, first in the case of the defocusing Nonlinear Schrödinger Equation (NLS). In 1975 Shabat observed that the inverse scattering problem for the one-dimensional Schrödinger equation could be rephrased as a RHP. Because of the connection between Schrödinger operators and the Korteweg-de Vries (KdV) equation, this meant that KdV, and by extension all 1+1-dimensional integrable systems, could be solved by a RHP. In the case of defocusing NLS, Shabat's observation amounts to the following (see e.g. [DZ4]). Let $q(x,t)$ be the solution of NLS on the line

$$\begin{cases} iq_t + q_{xx} - 2|q|^2 q = 0 \\ q(x, t=0) = q_0(x) \end{cases} \tag{57}$$

where $q_0(x) \to 0$ sufficiently rapidly as $|x| \to \infty$. Just as KdV is associated with the Schrödinger operator, NLS is associated with a first order, two-by-two scattering problem

$$\frac{d\psi}{dx} = i\frac{z}{2}\sigma_3\psi + \begin{pmatrix} 0 & iq \\ -i\bar{q} & 0 \end{pmatrix}\psi\ , \qquad -\infty < x < \infty \tag{58}$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the third Pauli matrix. Let $r = r(z)$ be the reflection coefficient for (58) with $q = q_0$. The map $\hat{R} : q \mapsto r$ is the analog for NLS of the OP maps $\hat{F}$ and $\hat{\phi}$. Now, for fixed x and t, let $m = m(z; x, t)$ be the solution of the normalized RHP $(\mathbb{R}, v_{x,t})$ where $\mathbb{R}$ is oriented from $-\infty$ to $+\infty$ and

$$\begin{cases} v_{x,t}(z) = \begin{pmatrix} 1 - |r(z)|^2 & re^{i\theta} \\ -\bar{r}e^{i\theta} & 1 \end{pmatrix} \\ \theta = xz - tz^2 \end{cases} , \qquad z \in \mathbb{R}. \tag{59}$$

Let $m_1(x,t)$ be the residue of m at $z = \infty$,

$$m(z; x, t) = I + \frac{m_1(x,t)}{z} + O\left(\frac{1}{z^2}\right)\ .$$

Then

$$q(x,t) = -i(m_1(x,t))_{12} \tag{60}$$

How does one prove (60)? At the functorial level, $\hat{R}$ is really a map from the category of differential operators to the category of RHP's,

$$L(q) \mapsto q \mapsto r \mapsto v_{x,t}$$

where $L(q) = i\sigma_3 \frac{d}{dx} + \begin{pmatrix} 0 & iq \\ -i\bar{q} & 0 \end{pmatrix}$, and so the key question becomes: "How is the differential operator encoded into the formalism of RHP's?"

To answer this question, observe that

$$\psi = \psi(z; x, t) \equiv m(z; x, t)e^{i\frac{\theta}{2}\sigma_3} \tag{61}$$

solves the RHP

- $\psi(z; x, t)$ analytic on $\mathbb{C}\backslash\mathbb{R}$
- $\psi_+ = \psi_- \begin{pmatrix} 1 - |r(z)|^2 & r(z) \\ -\overline{r(z)} & 1 \end{pmatrix}, \qquad z \in \mathbb{R}$

where the jump matrix is now independent of x and t. Differentiating with respect to x, we obtain

$$\psi_{x+} = \psi_{x-}\begin{pmatrix} 1-|r(z)|^2 & r(z) \\ -\overline{r(z)} & 1 \end{pmatrix}$$

from which it follows that $T \equiv \psi_x \psi^{-1}$ has no jump across $\mathbb{R}$, and hence is entire. But as $z \to \infty$,

$$\begin{aligned} T &= m_x m^{-1} + m\frac{iz}{2}\sigma_3 m^{-1} \\ &= iz\frac{\sigma_3}{2} + A + O\left(\frac{1}{z}\right) \end{aligned}$$

for some constant matrix A. By Liouville, we must then have $T = iz\frac{\sigma_3}{2} + A$ or

$$\psi_x = iz\frac{\sigma_3}{2}\psi + A\psi \tag{62}$$

Simple symmetry considerations imply that A is of the form $\begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix}$, and hence we recover the differential equation (58). Differentiating the ψ-RHP with respect to t yields similarly an equation of the form

$$\psi_t = B\psi \tag{63}$$

for some explicit matrix $B = B(z, q, q_x)$. Cross-differentiating (62) and (63), $(\psi_x)_t = (\psi_t)_x$, then yields the NLS equation (57). It is in this way in general that identities and differential relationships are encoded into the RHP.

To apply the above methodology to OP's, consider the solution $X^{(n)}$ of the RHP $(\mathbb{R}, v)$ above. Observing that $X^{(n+1)}$ satisfies the same jump relation as $X^{(n)}$ across $\mathbb{R}$, we conclude as before that $T \equiv X^{(n+1)}(X^{(n)})^{-1}$ is entire. But

$$\begin{aligned} T &= X^{(n+1)}(z)(X^{(n)}(z))^{-1} \\ &= \left[\left(I + \frac{X_1^{(n+1)}}{z} + O\left(\frac{1}{z^2}\right)\right) z^{(n+1)\sigma_3}\right]\left[\left(I + \frac{X_1^{(n)}}{z} + O\left(\frac{1}{z^2}\right)\right) z^{n\sigma_3}\right]^{-1} \\ &= z\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + X_1^{(n+1)}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} X_1^{(n)} + O\left(\frac{1}{z}\right) \end{aligned}$$

and again by Liouville we conclude that

$$X^{(n+1)}(z) = \left(z\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + X_1^{(n+1)}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} X_1^{(n)}\right) X^{(n)}(z) \tag{64}$$

from which the three-term recurrence relation (6) now follows by a simple computation. Moreover, if we replace the weight $\omega(x)$ with $\omega_t(x) = \frac{e^{2xt}\omega(x)}{\int_{\mathbb{R}} e^{2st}\omega(s)ds}$, then

$$W^{(n)}(z;t) \equiv X^{(n)}(z;t)e^{(tz+g(t))\sigma_3}\ , \qquad g(t) \equiv -\frac{1}{2}\log\int_{\mathbb{R}} e^{2st}\omega(s)ds\ , \tag{65}$$

solves the RHP $(\mathbb{R}, v)$ with jump matrix $v = \begin{pmatrix} 1 & \omega(x) \\ 0 & 1 \end{pmatrix}$ independent of t. Differentiating with respect to t, we obtain as above a differential equation for $W^{(n)}$

$$\frac{d}{dt}W^{(n)} = ((z+\dot{g})\sigma_3 + X_1^{(n)}\sigma_3 - \sigma_3 X_1^{(n)})W^{(n)}\,.$$

Using Γ to denote the shift operator, $\Gamma W^{(n)} = W^{(n+1)}$, equation (64) takes the form

$$\Gamma W^{(n)} = \left(z \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + X_1^{(n+1)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} X_1^{(n)} \right) W^{(n)} \tag{66}$$

Cross-"differentiating" (65) and (66), $\frac{d}{dt}\Gamma W^{(n)} = \Gamma \frac{dW^{(n)}}{dt}$, one is led immediately to the Toda flow (23).

In another direction, if $\omega(x) = e^{-V(x)}$, $V(x) = \gamma_m x^{2m} + \ldots$, $\gamma_m > 0$, then $U^{(n)} \equiv X^{(n)} e^{\frac{1}{2}V(x)\sigma_3}$ satisfies a jump relation across $\mathbb{R}$ with jump matrix $v = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which is independent of z, and by the above general methodology this leads to a differential equation for $U^{(n)}$ with respect to z, $\frac{dU^{(n)}}{dz} = DU^{(n)}$, for some explicit D. Cross-"differentiation", $\frac{d}{dz}\Gamma U^{(n)} = \Gamma \frac{dU^{(n)}}{dz}$, then leads to so-called "string equations" for the recurrence coefficients a_n, b_n.

Applying the above methodology to the RHP (S^1, v) for OP's on the unit circle, we obtain, in particular, simple and direct proofs of Szegö recurrence, Geronimus' Theorem on the Schur iterates, and the Pinter-Nevai formula (see [Sim2], and below). Indeed, let $Y^{(n)}$ solve the RHP (S^1, v) above. Then one observes that $V^{(n)} \equiv Y^{(n+1)} \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}$ satisfies the same jump relation as $Y^{(n)}$ across S^1,

$$V_+^{(n)} = V_-^{(n)} \begin{pmatrix} 1 & \omega z^{-n} \\ 0 & 1 \end{pmatrix},$$

and hence $V^{(n)}(Y^{(n)})^{-1}$ is entire. As before, this leads to an equation for $V^{(n)}$ and $Y^{(n)}$, which takes the form

$$Y^{(n+1)} \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} = V^{(n)} = \begin{pmatrix} z + \hat{a}_n & \hat{b}_n \\ \hat{c}_n & 1 \end{pmatrix} Y^{(n)} \tag{67}$$

for suitable constants $\hat{a}_n, \hat{b}_n, \hat{c}_n$. Furthermore $(\det Y^{(n)})_+ = (\det Y^{(n)})_- \det v = (\det Y^{(n)})_-$, and so $\det Y^{(n)}$ is entire. But $\det Y^{(n)} = \det \left(Y^{(n)} \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} \right) \to 1$ as $z \to \infty$, and hence $\det Y^{(n)} \equiv 1$. Taking determinants of both sides of (67), we find the relation

$$\hat{a}_n = \hat{b}_n \hat{c}_n \,. \tag{68}$$

¿From the first column of (67) we obtain the relations

$$\Phi_{n+1} = (z + \hat{a}_n)\Phi_n - \kappa_{n-1}^2 \hat{b}_n \Phi_{n-1}^* \tag{69}$$

$$-\kappa_n^2 \Phi_n^* = \hat{c}_n \Phi_n - \kappa_{n-1}^2 \Phi_{n-1}^* \,. \tag{70}$$

Eliminating Φ_{n-1}^*, we obtain the Szegö recurrence relation (8)

$$\Phi_{n+1} = z\Phi_n - \bar{\alpha}_n \Phi_n^* \tag{71}$$

with ϕ_n replaced by Φ_n, and with Verblunsky coefficient

$$\alpha_n = \kappa_n^2 \overline{\hat{b}_n} \,. \tag{72}$$

Letting $z \to \infty$ in (70) we find

$$\hat{c}_n = \kappa_n^2 \alpha_{n-1} \tag{73}$$

and hence by (68)

$$\hat{a}_n = \bar{\alpha}_n \alpha_{n-1}. \tag{74}$$

Now consider the second column in (67). Setting

$$r_n = C(\Phi_n \omega s^{-n}) \ , \qquad t_n = C(\Phi_n^* \omega s^{-n-1}) \tag{75}$$

and using (72), (73) and (74), we obtain as in (69) and (70)

$$z r_{n+1} = (z + \bar{\alpha}_n \alpha_{n-1}) r_n - \bar{\alpha}_n \left(\frac{\kappa_{n-1}}{\kappa_n} \right)^2 t_{n-1} \tag{76}$$

$$-z \kappa_n^2 t_n = \kappa_n^2 \alpha_{n-1} r_n - \kappa_{n-1}^2 t_{n-1} \, . \tag{77}$$

Eliminating t_{n-1} as we eliminated Φ_{n-1}^* above, (76) and (77) reduce to

$$r_{n+1} = r_n - \bar{\alpha}_n t_n \tag{78}$$

$$z t_{n+1} = -\alpha_n r_n + t_n \, . \tag{79}$$

Defining

$$f_n \equiv t_n / r_n \tag{80}$$

and using (78) and (79), we obtain the recurrence relation

$$z f_{n+1} = \frac{f_n - \alpha_n}{1 - \bar{\alpha}_n f_n} \ , \qquad n \geq 0. \tag{81}$$

In particular, for $z = 0$, we see that

$$\alpha_n = f_n(0) \tag{82}$$

and so (81) can be written in the form

$$z f_{n+1} = \frac{f_n - f_n(0)}{1 - \overline{f_n(0)} f_n} \ , \qquad n \geq 0. \tag{83}$$

Finally observe that

$$f_0(z) = \frac{t_0}{r_0} = \frac{\int_{S^1} \frac{\pi_0^*}{s-z} \omega \frac{ds}{2\pi i s}}{\int_{S^1} \frac{\pi_0}{s-z} \omega \frac{ds}{2\pi i}} = \frac{\int_{S_1} \frac{d\mu(\theta)}{s-z}}{\int_{S_1} s \frac{d\mu(\theta)}{s-z}} \ , \qquad s = e^{i\theta} \tag{84}$$

where $d\mu(\theta) = \omega(\theta) \dfrac{d\theta}{2\pi}$.

Geronimus' Theorem (see [Sim2]) states the following: Let

$$F(z) = \int_{S^1} \frac{s+z}{s-z} \, d\mu(\theta)$$

be the Carathéodory function for $d\mu$ and let $f_{\text{Schur}} \equiv \frac{1}{z} \frac{F(z)-1}{F(z)+1}$ be the associated Schur function. Let $(f_n)_{n \geq 0}$ solve the recurrence relation (83) with $f_n|_{n=0} = f_{\text{Schur}}$. Then

$$f_n(0) = \alpha_n \ , \qquad n \geq 0$$

where $\{\alpha_n\}_{n \geq 0}$ are the Verblunsky coefficients for $d\mu$.

However, a simple computation shows that $f_{\rm Schur}$ is precisely f_0 in (84): Hence, using the general methodology for RHP's as above, we have proved Geronimus' Theorem. Moreover we have the following formula for the Schur iterates:

$$f_n(z) = \frac{t_n}{r_n} = \frac{\int_{S^1} \frac{\Phi_n^* s^{-n}}{s-z} d\mu(\theta)}{\int_{S^1} \frac{\Phi_n s^{-n+1}}{s-z} d\mu(\theta)} \ , \qquad n \geq 0 \tag{85}$$

which reduces simply, using (3.2.52) and (2.2.53) [Sim2], to Golinskii's formula ([Sim2], Thm. 32.7).

Finally we note from [Sim2], (1.3.79), together with the simple identity $\int \frac{s}{s-z} d\mu = 1 + z \int \frac{d\mu}{s-z}$, that

$$f_n = \frac{f_{\rm Schur} B_{n-1} - A_{n-1}}{zB^*_{n-1} - zA^*_{n-1} f_{\rm Schur}} = \frac{(B_{n-1} - zA_{n-1}) \int \frac{d\mu}{s-z} - A_n}{(zB^*_{n-1} - A^*_{n-1}) \int \frac{sd\mu}{s-z} + A^*_{n-1}} \tag{86}$$

where A_{n-1}, B_{n-1} are the Wall polynomials. But from (85), we obtain

$$f_n(z) = \frac{z^n \int_{S^1} (\frac{\Phi_n^*(s)s^{-n} - \Phi_n^*(z) z^{-n}}{s-z}) d\mu(\theta) + \Phi_n^*(z) \int \frac{d\mu(\theta)}{s-z}}{z^n \int_{S^1} (\frac{\Phi_n(s)s^{-n} - \Phi_n(z) z^{-n}}{s-z}) s d\mu(\theta) + \Phi_n(z) \int s \frac{d\mu(\theta)}{s-z}} \, .$$

Comparing with (86) we obtain

$$\Phi_n^*(z) = B_{n-1} - zA_{n-1} \tag{87}$$

or equivalently

$$\Phi_n(z) = zB^*_{n-1} - A^*_{n-1} \tag{88}$$

which is the Pinter-Nevai formula (see [Sim2]) relating the OP's to the Wall polynomials.

In addition to the formulae and identities obtained above for OP's using the RHP's $(\mathbb{R}, v)$ and (S^1, v), one can, using RHP's closely related to $(\mathbb{R}, v)$ and (S^1, v), derive formulae for Toeplitz and Hankel determinants, or more precisely "relative" Toeplitz and Hankel determinants, that are particularly useful for asymptotic analysis. The asymptotic analysis of Toeplitz and Hankel determinants, dating back at least to the work of Szegö in 1915, is of considerable, and continuing, mathematical and physical interest, and we refer the reader to [BW], [E] and the references therein for more information and recent results. The "relative" determinant formulae are as follows.

Let $\omega_1(x), \omega_2(x) \geq 0$ be two weights on $\mathbb{R}$ and let $D_n(\omega_1\omega_2)$, $D_n(\omega_2)$ be the Hankel determinants associated with the measures $\omega_1(x)\omega_2(x)dx$ and $\omega_2(x)dx$ respectively. (Here we do not require $\omega_1\omega_2 dx$ and $\omega_2 dx$ to be probability measures.) Then

$$\log \frac{D_n(\omega_1\omega_2)}{D_n(\omega_2)} = \int_0^1 dt \int_{\mathbb{R}} R_t(x) \left(\frac{d}{dt} \log \omega_t(x) \right) dx \tag{89}$$

where $\omega_t = 1 - t + t\omega_1(x)$, $0 \leq t \leq 1$, and R_t is expressed in terms of the solution $X_t^{(n+1)} = ((X_t^{(n+1)})_{ij})_{1 \leq i,j \leq 2}$ of the RHP $(\mathbb{R}, v_t(x))$ in (39) with

$$v_t(x) = \begin{pmatrix} 1 & \omega_t(x)\omega_2(x) \\ 0 & 1 \end{pmatrix},$$

as follows:

$$R_t(x) = \frac{1}{2\pi i} \Big(\big(X_t^{(n+1)}\big)_{11} \big(X_t^{(n+1)}\big)'_{21} - \big(X_t^{(n+1)}\big)'_{11} \big(X_t^{(n+1)}\big)_{21} \Big) \omega_t \omega_2. \tag{90}$$

Similarly, if $\omega_1(\theta)$, $\omega_2(\theta) \geq 0$ are two weights on S^1, with associated Toeplitz determinants $\Delta_n(\omega_1\omega_2)$, $\Delta_n(\omega_2)$ respectively, then

$$\log \frac{\Delta_n(\omega_1\omega_2)}{\Delta_n(\omega_2)} = \int_0^1 dt \int_{S^1} R_t(\theta) \frac{d}{dt} \log \omega_t(\theta) \frac{d\theta}{2\pi} \tag{91}$$

where $\omega_t(\theta) = 1 - t + t\omega_1(\theta)$, $0 \leq t \leq 1$, and $R_t(\theta)$ is expressed in terms of the solution $Y_t^{(n+1)} = ((Y_t^{(n+1)})_{ij})_{1 \leq i,j \leq 2}$ of the RHP $(S^1, v_t(\theta))$ in (50) with

$$v_t(\theta) = \begin{pmatrix} 1 & \omega_t(\theta)\omega_2(\theta) z^{-(n+1)} \\ 0 & 1 \end{pmatrix}, \quad z = e^{i\theta},$$

as follows:

$$R_t(\theta) = \Big(\big(Y_t^{(n+1)}\big)_{11} \big(Y_t^{(n+1)}\big)'_{21} - \big(Y_t^{(n+1)}\big)'_{11} \big(Y_t^{(n+1)}\big)_{21} \Big) \frac{\omega_t \omega_2}{z^n} \tag{92}$$

where $\prime \equiv \frac{d}{dz}$.

The functions $R_t(x)$, $R_t(\theta)$ have the interpretation as 1-point functions

$$R_t(x) = (n+1) \int_{x_i \in \mathbb{R}, 1 \leq i \leq n} d\mu(x, x_1, x_2, \ldots, x_n) \tag{93}$$

$$R_t(\theta) = (n+1) \int_{\theta_i \in S^1, 1 \leq i \leq n} d\mu(\theta, \theta_1, \ldots, \theta_n) \tag{94}$$

for the random particle ensembles (see [Meh]) with distributions

$$d\mu(x_0, x_1, \ldots, x_n) = (1/Z_{\mathbb{R}}) \prod_{0 \leq j < k \leq n} (x_i - x_j)^2 \prod_{j=0}^{n} (\omega_t\omega_2)(x_j) dx_0 dx_1 \ldots dx_n \tag{95}$$

and

$$d\mu(\theta_0, \theta_1, \ldots, \theta_n) = (1/Z_{S^1}) \prod_{0 \leq i < j \leq n} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=0}^{n} (\omega_t\omega_2)(\theta_j) d\theta_0 d\theta_1 \ldots d\theta_n \tag{96}$$

where $Z_{\mathbb{R}}$, Z_{S^1} are normalization constants.

Note that on S^1 we can set $\omega_2 = 1$, so that $\Delta_n(\omega_2) = 1$ and (91) gives us a formula, first derived in [D1], purely for $\Delta_n(\omega_1)$. In the non-compact situation on $\mathbb{R}$, this clearly cannot be done and we must always work with relative determinants as in (89).

Formulae (89),(91) are due to Deift [D3], and may be proved by generalizing the proof of (91) given in [D1] for the case $\omega_2 = 1$. A key ingredient in the proof is the notion of an *integrable operator*: If Σ is an oriented contour in $\mathbb{C}$, we say that an operator K acting on $L^p(\Sigma)$, $1 < p < \infty$, is *integrable* if it has a kernel of the form

$$K(z, z') = \frac{\sum_{j=1}^{\ell} f_j(z) g_j(z')}{z - z'}, \qquad z, z' \in \Sigma \tag{97}$$

for some functions $f_j, g_k \in L^\infty(\Sigma)$, $1 \leq j, k \leq \ell$. Special examples of integrable operators appeared in the 1960's in the work of McCoy, Tracy and others, and elements of the general theory were discovered by Sakhnovich in the late 60's, but the full general theory of such operators is due to Its, Izegin, Korepin and Slavnov [IIKS] in 1990. Integrable operators have many useful properties (see e.g. [D1]). In particular, if K is integrable as in (97) above, then so is $(1-K)^{-1} - 1$,

$$(1-K)^{-1} = 1 + \frac{\sum_{j=1}^{\ell} F_j(z) G_j(z')}{z - z'}$$

for suitable F_j, G_k, $1 \le j, k \le l$. Furthermore, quite remarkably, the functions $F = (F_1, \ldots, F_l)^T$, $G = (G_1, \ldots G_l)^T$ can be computed in terms of the solution of a canonical, auxiliary RHP. Indeed, define the jump matrix $v = I - 2\pi i f g^T$ on Σ, where $f = (f_1, \ldots, f_l)^T$, $g = (g_1, \ldots, g_l)^T$, and assume for simplicity that $\sum_{j=1}^{\ell} f_j(z) g_j(z) = 0$, $z \in \Sigma$. Then, if m solves the normalized RHP (Σ, v), we have

$$F = m_\pm f \quad \text{and} \quad G = (m_\pm^T)^{-1} g. \tag{98}$$

The proofs of (89) and (91) proceed by expressing the relative determinants $\frac{D_n(\omega_1\omega_2)}{D_n(\omega_2)}$, $\frac{\Delta_n(\omega_1\omega_2)}{\Delta_n(\omega_2)}$ in terms of Fredholm determinants of integrable operators K,

$$\begin{aligned} \log\det(1-K) &= \int_0^1 \frac{d}{dt} \log\det(1-tK) \\ &= -\int_0^1 \operatorname{tr}\left(\frac{1}{1-tK} K\right) dt \end{aligned}$$

and then using (98) to express $(1-tK)^{-1}K = ((1-tK)^{-1} - 1)/t$ in terms of the solution of the auxiliary RHP associated to tK. We shall say more about (89) and (91) in what follows.

3. Applications of $(\mathbb{R}, v)$ and (S^1, v): asymptotics

In this section we consider the asymptotics of OP's, denoted (b) in Section 2. In Section 2, the goal was to show how a variety of identities, equations and formulae, mostly classical and well-known, follow from a single, basic methodology in RHP's. Here the goal is to describe new results on the asymptotics of OP's that follow from the RH method, utilizing in particular the non-linear, non-commutative, steepest descent method introduced in [DZ1] in 1993. Although much was known (see [Sze]) about the detailed asymptotic behavior of classical OP's, like Hermite, Laguerre, Jacobi polynomials, etc., both on and off the contour of orthogonality, little was known about the detailed asymptotics of OP's with respect to general weights. The main tool that makes possible the detailed analysis of the asymptotics of classical OP's is the existence of integral representations for these polynomials, to which the classical method of steepest descent can be applied (see, for example, [Sze], Section 8.71). For general weights, one may view the RHP's $(\mathbb{R}, v(x))$ and $(S^1, v(\theta))$ as non-commutative analogs of these integral representations, with the non-commutative steepest descent method now playing the role of the classical steepest descent method.

We now describe the steepest descent method for RHP's in broad outline: Unfortunately we do not have sufficient space in this article to describe the method in detail. In the case of NLS (cf. (59) and (60)), we write the solution $q(x,t)$ of the Cauchy problem for NLS as a functional f, say, of the data $re^{i\theta}$,

$$q(x,t) = f(re^{i\theta}). \tag{99}$$

¿From (35) and (60) we see that

$$f(re^{i\theta}) = \left(\int_{\mathbb{R}} \mu(s;x,t)(\omega_+ + \omega_-)\frac{ds}{2\pi}\right)_{12}. \tag{100}$$

Using the factorization

$$v_{x,t} = \begin{pmatrix} 1 & -re^{i\theta} \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ -\bar{r}e^{-\theta} & 1 \end{pmatrix} \tag{101}$$

(cf. (27)), so that

$$w_+ = \begin{pmatrix} 0 & 0 \\ -\bar{r}e^{-\theta} & 0 \end{pmatrix}, \qquad w_- = \begin{pmatrix} 0 & -\bar{r}e^{-i\theta} \\ 0 & 0 \end{pmatrix} \tag{102}$$

we obtain

$$q(x,t) = \left(\int_{\mathbb{R}} ((I - C_\omega^{\mathbb{R}})^{-1} I)(\omega_+ + \omega_-) \frac{ds}{2\pi} \right)_{12}. \tag{103}$$

For r "small", we have

$$\begin{aligned} q(x,t) &= \left(\int_{\mathbb{R}} ((I + \text{"small"})(\omega_+ + \omega_-) \frac{ds}{2\pi} \right)_{12} \\ &= \int_{\mathbb{R}} r(s) e^{i(xs - ts^2)} \frac{ds}{2\pi} + \text{"small"} \end{aligned}$$

indicating that the classical steepest descent method can be applied as $t \to \infty$. However, when r is no longer "small", we see from the non-linear dependence of $q(x,t)$ on r in (103), and from the matrix nature of the problem, that a non-linear, non-commutative version of the steepest descent method is required, and this is the kind of method that was introduced in [DZ1]. In the classical steepest descent method, the integral localizes as $t \to \infty$ to a small neighborhood of the stationary phase point(s), $\theta'(z_0) = 0$, $z_0 = x/2t$ in the case of NLS, and an explicit asymptotic formula for the solution is then obtained by evaluating a Gaussian integral: in the fully non-linear case (see [DZ2] [DZ4]) the RHP $(\mathbb{R}, v_{x,t})$ localizes to a RHP in the neighborhood of the stationary phase point $z_0 = x/2t$, and an asymptotic form for the solution

$$q(x,t) \sim \frac{1}{t^{1/2}} \alpha(z_0) e^{i(tz_0^2 - \beta(z_0) \log t)} \tag{104}$$

is then obtained by solving this local RHP explicitly (in terms of parabolic cylinder functions, as it turns out). The asymptotic form (104) was first obtained by Zakharov and Manakov [ZM], by other means. In situations where there is more than one stationary phase point, for example for MKdV, where $\theta = xz + 4tz^3$ with stationary phase points $\pm z_0 = \pm\sqrt{-x/12t}$, the long-time behavior of solutions of MKdV (see [DZ1]) is a superposition of NLS-like contributions from $+z_0$ and $-z_0$, as long as these points remain separated, i.e. $\frac{-x}{t} > c > 0$. However, in the space-time region where $-x/12t \to 0$, and hence $+z_0 \to -z_0$, one is in a non-linear "caustic" region which is manifested by the solution taking the form of a self-similar oscillation, $q(x,t) \sim \dfrac{1}{(3t)^{1/3}} u(x/(3t)^{1/3})$, where u is a solution of the Painlevé II equation $u''(t) = tu + 2u^3$ (see [DZ1]).

Up till this point, the RH asymptotic theory proceeded as a non-linear analog of the classical steepest descent method in which all the phenomena that arose could be viewed as non-linear counterparts of phenomena that had already arisen in the linear, scalar situation. However, with the analysis of the collisionless shock region for KdV (see [DVZ1],[DZ3]), and the analysis of the asymptotic behavior of solutions of the Painlevé II equation, it began to be clear that there were phenomena

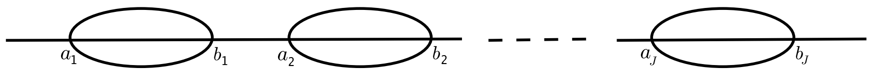

FIGURE 2. The contour $\hat{\Sigma}$

inherent in the non-linear steepest descent method that had no analog in the classical situation. Most importantly, it became clear that instead of stationary phase points, one could have "stationary phase lines" in which case all the points on some interval in $\mathbb{C}$ contributed equally to the asymptotic behavior of the solution of the problem. Moreover, in place of modulated linear oscillations as in (104), one would now have genuinely non-linear oscillations described in terms of Jacobi's *sn* and *cn* functions, etc. A systematic extension of the steepest descent method to allow for such "stationary phase lines" and genuinely non-linear oscillations was presented by Deift, Venakides and Zhou [DVZ2] in the context of their work on the zero dispersion problem. Soon thereafter, using the methods in [DVZ2] together with recent developments in the theory of logarithmic potentials with external fields (see [ST], and also [DKM]), the authors in [DKMVZ2] derived so-called Plancherel-Rotach asymptotics for OP's with measures of the form

$$e^{-V(x)}dx, \quad V(x) = \gamma x^{2m} + \delta x^{2m-1} + \cdots \quad \gamma > 0, \tag{105}$$

and in [DKMVZ1], for measures of the form

$$e^{-nQ(x)}dx, \quad Q(x)/\log|x| \to +\infty \quad \text{as} \quad |x| \to \infty, \tag{106}$$

where $Q(x)$ is real analytic on $\mathbb{R}$. As described in [DKMVZ2] one obtains as $n \to \infty$ precise pointwise asymptotics for the OP's $P_n(z)$ for all $z \in \mathbb{C}$, as well as detailed asymptotics for a_n, b_n, γ_n and the zeros of $p_n(z)$. In the special case $e^{-n(x^4 - tx^2)}dx$, Bleher and Its [BI] obtained asymptotics for the associated OP's using RH techniques and a mixture of steepest descent/isomonodromy ideas.

In broad outline the method proceeds as follows. For weights $e^{-V(x)}$ as above one first scales $x \to xn^{1/2m}$ so that $e^{-V(x)} \to e^{-nV_n(x)}$, where $V_n(x) = \gamma x^{2m} + \frac{\delta}{n^{1/2m}}x^{2m-1} + \cdots$. Next, one considers the so-called equilibrium measure $d\mu_{\text{eq}}$ for the logarithmic potential problem associated with OP's (see [ST]). By [DKM], for weights $e^{-nV_n(x)}$ or $e^{-nQ(x)}$ as above, $d\mu_{\text{eq}}$ is supported on a finite union of disjoint intervals $\cup_{i=1}^{J}(a_i, b_i)$, $J < \infty$ (in the case $e^{-nV_n(x)}$, $J = 1$). Next one introduces the so-called "g" function, $g(z) \equiv \int_{\mathbb{R}} \log(z-s)d\mu_{\text{eq}}(s) \sim \log z$ as $z \to \infty$. Along with $d\mu_{\text{eq}}$, the logarithmic potential problem also produces a Lagrange multiplier ℓ, and we set $\tilde{X}^{(n)} \equiv e^{\frac{n\ell}{2}\sigma_3}X^{(n)}(z)e^{-ng(z)\sigma_3}e^{-\frac{n\ell}{2}\sigma_3}$. One observes that $\tilde{X}^{(n)}$ now solves a *normalized* RHP $(\mathbb{R}, \tilde{v})$ for some explicit jump matrix $\tilde{v}$. In the key step, the RHP for $\tilde{X}^{(n)}$ is now deformed to a RHP on a contour $\hat{\Sigma}$ of the form shown in Figure 2. By the properties of $g(z)$, or more properly, the properties of $d\mu_{\text{eq}}$, it turns out that as $n \to \infty$, $\hat{v}$, the jump matrix for the deformed RHP on $\hat{\Sigma}$, converges

$$\hat{v}(z) \to I \tag{107}$$

exponentially for all $z \in \hat{\Sigma} \setminus \cup_{i=1}^{J} [a_i, b_i]$. Thus as $n \to \infty$, the RHP reduces to a limiting RHP on the union of intervals $\cup_{i=1}^{J}[a_i, b_i]$. On each of the intervals (a_i, b_i),

$\hat{v}(z)$ has the simple form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and this limiting RHP can be solved explicitly in terms of the function theory on the hyper-elliptic Riemann surface obtained by gluing together two copies of $\mathbb{C}\setminus \cup_{i=1}^{J}(a_i, b_i)$ in the standard way. However, the convergence rate in (107) is not uniform, becoming slower and slower as z approaches the end points a_i, b_i. The natural topology for RHP's is convergence for the coefficients of $\hat{v}$ in $L^p \cap L^\infty(\hat{\Sigma})$ (cf. (30)(33)), and the lack of uniform convergence in (107) constitutes the major technical difficulty in implementing the steepest descent method as described above. We refer the reader to [DKMVZ1, DKMVZ2] for more details.

We now consider the relative determinant formulae (89) and (91) and their associated RHP's $(\mathbb{R}, v_t(x))$ and $(S^1, v_t(\theta))$ respectively. The celebrated strong Szegö limit theorem, in the definitive form due to Ibragimov (see [Sim2] for many proofs and much historical discussion) states that if $d\mu(\theta) = e^{-V(\theta)}\dfrac{d\theta}{2\pi}$, and $V(\theta)$ has Fourier coefficients $\{\hat{V}_k\}$ satisfying $\sum_{k=1}^{\infty} k|\hat{V}_k|^2 < \infty$, then as $n \to \infty$

$$\ln \Delta_n(e^{-V}) = (n+1)\hat{V}_0 + \sum_{k=1}^{\infty} k|\hat{V}_k|^2 + o(1) \ . \tag{108}$$

In addition to the many proofs in [Sim2], (108) can also be proved, under certain additional smoothness assumptions on $V(\theta)$, by applying the steepest descent method to the RHP's $(S^1, v_t(\theta))$, $0 < t < 1$. The situation is simpler than in [DKMVZ1, DKMVZ2], but the argument in this situation is particularly illustrative of the emergence of a "stationary phase line": details are given in [D1]. There is also a version of the strong Szegö limit theorem for block Toeplitz determinants (see [W1], [W2], and also [Bot] for more recent results). In the block Toeplitz case, the analog of (108) contains a certain Fredholm determinant which is difficult to evaluate in elementary terms. In certain cases the method in [D1] extends to the block Toeplitz case and, quite surprisingly, the term corresponding to this Fredholm determinant is evaluated automatically (see [IJK]).

In [BCW1] (see also [BCW2]) the authors state the following analog of the Szegö strong limit theorem for the case of Hankel matrices. Let $\omega_2 = e^{-x^2}$ and let $\omega_1(x) > 0$ have the property that $\omega_1(x) \to 1$ sufficiently rapidly as $|x| \to \infty$. Then as $n \to \infty$

$$\ln \frac{D_n(\omega_1\omega_2)}{D_n(\omega_2)} = \frac{\sqrt{2(n+1)}}{\pi}\int_{\mathbb{R}} \log \omega_1(x)dx + \frac{1}{4\pi}\int_{\mathbb{R}} |k| \ |\hat{f}(k)|^2 dk + o(1) \tag{109}$$

where $\hat{f}(k) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int (\log \omega_1(x)) e^{-ikx}dx$. Using (89), this result can also be proved ([D4]) using the steepest descent method, not only for $\omega_2 = e^{-x^2}$, but also for more general weights, $\omega_2 = e^{-V(x)}$, $V(x) = \gamma x^{2m} + \ldots$, $\gamma > 0$, as above.

Riemann-Hilbert techniques and the RH method are useful not only for asymptotic evaluation, but also for estimation. For example, let $\omega(\theta) \in L^\infty(S^1)$ be a bounded weight on S^1 with Fourier coefficients $\omega_k = \displaystyle\int_{-\pi}^{\pi} e^{-ik\theta}\omega(\theta)\frac{d\theta}{2\pi}$, $k \in \mathbb{Z}$. Let $((T(\omega))_{jk})_{j,k=0}^{\infty} = (\omega_{j-k})_{j,k=0}^{\infty}$ denote the Toeplitz matrix associated with ω acting

on $\ell_2^+ = \left\{ u = (u_0, u_1, \ldots) : \sum_{k=0}^{\infty} |u_k|^2 < \infty \right\}$, and let $((T_n(\omega))_{jk})_{j,k=0}^n = (\omega_{j-k})_{j,k=0}^n$ denote the leading $(n+1) \times (n+1)$ section of $T(\omega)$. If ω is in the Wiener space $W^0 = \left\{ \omega : \sum_{j=-\infty}^{\infty} |\omega_j| < \infty \right\}$ with $\omega(\theta) > 0$, then, by a well known theorem of Krein, $(T(\omega))^{-1}$ exists as a bounded operator in ℓ_2^+. The question is the following: How closely does $(T_n(\omega))^{-1}$ approximate $(T(\omega))^{-1}$ for n large? Let $\nu = (\nu_k)_{k\in\mathbb{Z}}$ be a Beurling weight (see e.g. [Sim2]): Thus $\nu_k \geq 1$, $\nu_k = \nu_{-k}$ and $\nu_{j+k} \leq \nu_j \nu_k$ for all $j, k \in \mathbb{Z}$. In particular, $((1+|k|)^\ell)_{k\in\mathbb{Z}}$, $\ell > 0$, and $(e^{\alpha|k|})_{k\in\mathbb{Z}}$, $\alpha > 0$, are Beurling weights. Define the Beurling class

$$W_\nu = \left\{ \omega \in L^1(S^1) : \sum_{j\in\mathbb{Z}} \nu_j |\omega_j| < \infty \right\}.$$

Clearly $W_\nu \subset W^0$ for any Beurling weight ν. Let $\omega \in W_\nu$ for some ν and assume in addition, for simplicity, that the weights increase on $\mathbb{Z}_+$, i.e. $\nu_j \leq \nu_k$ for $0 \leq j < k$. Then the following is true [DO]: for n sufficiently large and $0 \leq j,\ k \leq n$,

$$\left| (T_n(\omega))^{-1}_{jk} - (T(\omega))^{-1}_{jk} \right| \leq c_\nu(\omega) \min(\nu^{-1}_{n+1-k}, \nu^{-1}_{n+1-j}) \tag{110}$$

for some constant $c_\nu(\omega)$. Thus for $0 \leq j, k \leq n$, $\big(T_n(\omega)\big)^{-1}_{jk}$ is a good approximation to $\big(T(\omega)\big)^{-1}_{jk}$, apart from the lower right corner $j \sim k \sim n$. This estimate is a generalization of an earlier estimate due essentially to Widom (see [BS] for references and further discussion). The proof of (110) in [DO] uses RH techniques in an essential way closely related to the proof of (86). The paper also contains other results for orthogonal polynomials on the unit circle, including a new RH proof of the reverse statement in Baxter's theorem (cf. [Sim2]). Interestingly, the Borodin-Okounkov operator [BO], or more properly, the Borodin-Okounkov-Case-Geronimo operator, which has emerged recently as a powerful tool in the analysis of Toeplitz determinants, arises naturally in the analysis in [DO].

The steepest descent method for varying weights $\omega(x) = e^{-nQ(x)}$ in [DKMVZ1] can also be applied to orthogonal polynomials on the unit circle with varying weights $\omega(\theta) = e^{-nQ(\theta)}$. For example in their analysis of the length $l_n = l_n(\pi)$ of the longest increasing subsequence of a random permutation π on n letters, the authors in [BDJ] prove that

$$\lim_{n\to\infty} \operatorname{Prob}\left(\frac{l_n - 2\sqrt{n}}{n^{1/6}} \leq t \right) = F_2(t) \tag{111}$$

where $F_2(t)$ is the Tracy-Widom distribution function for the largest eigenvalue of a random matrix from the Gaussian Unitary Ensemble. The proof of (111) in [BDJ] reduces, by a formula of Gessel, to the analysis of the Toeplitz determinant $\Delta_{n-1}(e^{s\cos\theta})$ where $s = (n+1)\left(1 - \frac{t}{2^{1/3}(n+1)^{2/3}}\right)$ as $n \to \infty$, and where t is the same as in (111). As indicated above, the method of [BDJ] is modeled on the RH steepest descent method in [DKMVZ1]. The same RH problem with weight $e^{s\cos\theta}$ on S^1 also appears in the work of Baik and Rains [BR] in their analysis of monotone subsequences of involutions.

The steepest descent method for OP's $\{\phi_n\}$ on the unit circle can also be used to obtain detailed information on the zeroes of the ϕ_n's as $n \to \infty$ (see [MMS]).

In a further development, the authors of [MM] have introduced an extension of the steepest descent method to non-analytic weights, obtaining in particular new results for the zeros of OP's on the unit circle for such weights.

Throughout this paper we have restricted our attention to measures that are smooth as in (37) and (47). The OP problem for general measures $d\mu$ is then analyzed (cf. **Technical Remark** above) by approximating the measure appropriately by smooth measures $d\mu_\epsilon$, and then taking the limit as $\epsilon \to 0$. This approach works well for the derivation of equations, formulae, etc., but for asymptotic questions one clearly needs a different approach. Recently remarkable connections have been discovered ([J]) between various combinatorial problems - random growth models, random word problems, tiling problems - and certain polynomials orthogonal with respect to discrete measures. The polynomials that arise include the classical Meixner, Charlier, Krawtchouk and Hahn polynomials (see [Sze]). Related discoveries have also been made in the representation of the infinite dimensional symmetric and unitary groups [BO1][BO2]. The Meixner, Charlier and Krawtchouk polynomials all have convenient integral representations (see [Sze]) and their asymptotic behavior can be read off using the classical method of steepest descent. This is unfortunately not the case for the Hahn polynomials (such polynomials are needed in particular to describe the tiling of hexagons by rhombi). It turns out, however, that discrete OP problems can be rephrased in terms of a discrete RHP, which is an analogue of the continuous case, and which was introduced by Borodin, along with a theory of discrete integrable operators, in [B]. In a significant further development of the nonlinear steepest descent method, the authors in [BKMM] extended the method to a wide class of discrete RHP's which includes the discrete RHP for the Hahn polynomials (as well as the other three discrete OP systems mentioned above). The relevant limit here is when the order of the OP's p_n becomes large and simultaneously the spacing between the points in the measures goes to zero at a prescribed rate (see [BKMM]). In this way the authors are able to analyze the Hahn polynomials asymptotically, proving *en route* a conjecture of Johansson in [J] that for hexagonal tiling the so-called "arctic circle" of [CLP] exhibits Tracy-Widom fluctuations as in (111) above. In [BO2] the authors also consider an asymptotic problem for Hahn polynomials using a discrete RHP, but the relevant limit is different from that in [BKMM].

Many researchers are currently involved in the application of RH techniques to the theory of OP's. In addition to those mentioned above, the list includes Chen, Claeys, Kapaev, Kitaev, Kuijlaars, van Assche and Vanlessen, amongst many others. Because of space limitations, however, we unfortunately cannot describe their work in any detail, and we must refer the reader to the literature.

4. Related areas

In this final section we will describe, very briefly, various areas related to OP's in which the RH method plays a role.

We first consider random matrix theory (RMT), which has been a major source of questions and challenges to OP theorists for over 40 years (see e.g. [Meh] and [D2]). The situation is as follows. A Unitary Ensemble (UE) is an ensemble of $N \times N$ Hermitian matrices $\{M = M^*\}$ with probability distribution

$$P_N(M)\, dM = \frac{1}{Z_N} e^{-\operatorname{tr} W(M)} dM \tag{112}$$

where

- dM denotes Lebesgue measure on the algebraically independent elements of M.
- $W(x)$ is a real-valued function that goes to $+\infty$ as $|x| \to \infty$. The case $W(x) = x^2$ gives rise to the Gaussian Unitary Ensemble (GUE).
- Z_N is a normalization coefficient.

"Unitary" refers to the fact that the distribution (112) is invariant under unitary conjugation, $M \to UMU^*$, U unitary. The Universality Conjecture for UE's (see [Meh] and [D2]) states, in particular, the following: Given W, if $J_N = c_N + s_N(-t,t)$ is a suitably centered and scaled interval in $\mathbb{R}$, then as $N \to \infty$, $P(J_N) = \mathrm{Prob}(M : M$ has no eigenvalues in $J_N)$ converges to a *universal limit* independent of W,

$$\lim_{N\to\infty} P(J_N) = \det(1 - S_t) \tag{113}$$

where S_t is the trace class operator with kernel $S_t(x,y) = \frac{\sin \pi(x-y)}{\pi(x-y)}$ acting in $L^2(-t,t)$. The specific form of the weight $e^{-W(x)}\,dx$ is reflected only in the precise values of c_N and s_N. OP's enter the picture because of the celebrated result of Gaudin and Mehta (see [Meh]) that if $B \subset \mathbb{R}$ is a Borel set, then

$$\mathrm{Prob}(M : M \,\text{has no eigenvalues in}\, B) = \det(1 - K_{N,B}) \tag{114}$$

where $K_{N,B}$ is the finite rank operator with kernel

$$K_N(x,y) = \sum_{j=0}^{N-1} p_j(x) p_j(y) e^{-\frac{1}{2}W(x)} e^{-\frac{1}{2}W(y)} \tag{115}$$

acting on $L^2(B)$, and $\{p_j\}_{j\geq 0}$ are the orthonormal polynomials (3) with respect to the weight $e^{-W(x)}\,dx$. Hence the question of proving universality as in (113) becomes a question of deriving the appropriate asymptotics for OP's, and this is the main scientific content of [DKMVZ2], [DKMVZ1], [D2] and [BI]. Of course, if the weight $e^{-W(x)}\,dx$ is classical, e.g. $W(x) = x^2$, and the asymptotics of the associated polynomials $\{p_j\}_{j\geq 0}$ can be derived from an integral representation, then universality for these ensembles can be proved without recourse to the RH steepest descent method, and this has been done by various authors (see [DKMVZ2], [DKMVZ1] for references to the literature).

Orthogonal ensembles (OE's) of $N \times N$ real symmetric matrices $\{M = \bar{M} = M^T\}$ and Symplectic Ensembles (SE's) of $2N \times 2N$ Hermitian self-dual matrices $\{M = M^*, JMJ^T = M^T\}$, where $J = \mathrm{diag}(\tau, \ldots, \tau)$, $\tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, equipped with invariant weights analogous to (112), are more difficult to analyze. Firstly, in the place of determinantal expressions as in (114), one obtains Pfaffians (see [Meh] for classical ensembles, [TW] for the general case)

$$\mathrm{Prob}(M : M \,\text{has no eigenvalues in}\, B) = \left(\det(1 - \hat{K}_{N,B})\right)^{1/2}, \tag{116}$$

and, moreover, the operators $\hat{K}_{N,B}$ are now 2×2 matrix operators with kernels $\left(\hat{K}_{N,ij}(x,y)\right)_{1\leq i,j\leq 2}$, $x, y \in B$. In contrast to (115), these kernels are most naturally expressed in terms of certain skew-orthogonal polynomials (see [Meh]), but for general weights $e^{-W(x)}\,dx$ the asymptotic behavior of such polynomials is not known. However Widom [W3] has shown that if W'/W is rational, then $\left(\hat{K}_{N,ij}(x,y)\right)$ can be expressed conveniently in terms of the orthonormal polynomials $\{p_j\}_{j\geq 0}$ with

respect to the weight $e^{-W(x)}\,dx$, so again, as in the unitary case, the question of universality of OE's and SE's becomes a question of analyzing the asymptotic behavior of OP's. The expressions for $\big(\hat{K}_{N,ij}(x,y)\big)_{1\le i,j\le 2}$ are now more cumbersome than (115) and significant new technical issues arise, but nevertheless, using the asymptotic analysis in [DKMVZ2] as a basic ingredient, it is indeed possible to use Widom's formulae in [W3] to prove universality for OE's and SE's with weights of the form $e^{-V(x)}\,dx$, $V(x) = \gamma x^{2m} + \cdots$, $\gamma > 0$. This is the content of [DG1] and [DG2].

Biorthogonal polynomials $\pi_k(x) = x^k + \cdots$, $\sigma_j(y) = y^j + \cdots$, $k, j \ge 0$,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \pi_k(x)\sigma_j(y) e^{-V(x)-W(y)+2\tau xy}\,dxdy = 0 \quad \text{if}\, j \ne k, \tag{117}$$

arise in the analysis of the theory of coupled random matrices. Here $V(x)$ and $W(y)$ grow sufficiently rapidly as $|x|, |y| \to \infty$, and $\tau \ne 0$. Various RH problems have been proposed to analyze these polynomials (see, in particular, [BEH], [K], [KM] and the references therein), but the analysis of the asymptotic behavior of these RHP's is still at a preliminary stage.

For $m \ge 2$, let $n = (n_1, n_2, \ldots, n_m)$ be a vector of non-negative integers, and let $\omega_1(x) \ge 0, \ldots, \omega_m(x) \ge 0$ be weights on $\mathbb{R}$ with finite moments. Let $|n| = n_1 + \cdots + n_m$. Multiple orthogonal polynomials (see [Apt]) of type I are polynomials $A_n^{(k)}$ for $k = 1, 2, \ldots, m$, $\deg A_n^{(k)} \le n_k - 1$ such that the function

$$H_n(x) = \sum_{k=1}^{m} A_n^{(k)}(x)\omega_k(x)$$

satisfies

$$\int_{\mathbb{R}} x^j H_n(x)\,dx = \begin{cases} 0, & \text{for}\, j = 0, \ldots, |n| - 2; \\ 1, & \text{for}\, j = |n| - 1. \end{cases} \tag{118}$$

Multiple orthogonal polynomials $L_n(x)$ of type II are monic polynomials of degree $|n|$ satisfying

$$\int_{\mathbb{R}} L_n(x) x^k \omega_j(x)\,dx = 0 \quad \text{for}\; k = 0, \ldots, n_j - 1,\; j = 1, \ldots, m. \tag{119}$$

Multiple orthogonal polynomials were first introduced by Hermite in his proof of the transcendence of e. In 2000, van Assche, Geronimo and Kuijlaars [vAGK] showed that multiple orthogonal polynomial problems of types I and II could be rephrased as RHP's analogous to the RHP of Fokas, Its and Kitaev for ordinary OP's, and they used these RHP's to derive various properties and relations for the multiple OP's. In the last year or two significant progress has been made in extending and applying the steepest descent method to RHP's which arise from multiple OP's in special cases. We mention, in particular, [BK], [ABK] and [KVaW], [KSVaW] and the references therein: In the first two papers the authors consider a random matrix ensemble $P_N(M)\,dM = \frac{1}{Z_N} e^{-N\,\mathrm{tr}(\frac{1}{2}M^2 - AM)}\,dM$, with external source A, first analyzed by Pastur, Brézin-Hikami, and later by Zinn-Justin. Under certain conditions on A, they show that the ensemble can be analyzed as $N \to \infty$ in terms of a 3×3 RHP to which an extension of the nonlinear steepest descent method can be applied: A new phenomenon now occurs in the analysis, which the authors term a "global opening of lenses" (see [ABK]). In the second two papers the authors analyze type I and type II Hermite-Padé approximations to the

exponential function, which they are again able to control by applying an extension of the steepest descent to a 3×3 RHP.

Riemann-Hilbert techniques can also be used to analyze the asymptotics of so-called *orthogonal Laurent polynomials*. Such polynomials arise in the following way. Let $V(x)$ be a real-analytic function on $\mathbb{R} \setminus \{0\}$ with the property

$$\lim_{|x|\to\infty} \frac{V(x)}{\ln|x|} = \lim_{|x|\to 0} \frac{V(x)}{\ln(|x|^{-1})} = +\infty.$$

Orthogonalization of the ordered basis $\{1, z^{-1}, z, z^{-2}, z^2, \ldots\}$ with respect to the pairing $(f,g) \mapsto \int_{\mathbb{R}} f(s)g(s)e^{-NV(s)}\,ds$ leads to the even degree and odd degree orthonormal Laurent polynomials $\{\phi_m\}_{m\geq 0}$: $\phi_{2n}(z) = \xi^{(2n)}_{-n}z^{-n} + \cdots + \xi^{(2n)}_{n}z^n$, $\xi^{(2n)}_{n} > 0$, $\phi_{2n+1}(z) = \xi^{(2n+1)}_{-n-1}z^{-n-1} + \cdots + \xi^{(2n+1)}_{n}z^n$, $\xi^{(2n+1)}_{-n-1} > 0$. Recently, McLaughlin, Vartanian and Zhou (see [MVZ] and the references therein) have used RHP-steepest descent methods to analyze the asymptotic behavior of the Laurent polynomials $\phi_{2n}(z), \phi_{2n+1}(z)$ and their associated norming constants $\xi^{(2n)}_{n}, \xi^{(2n+1)}_{-n-1}$ in the limit as $N \to \infty$, $N/n \to 1$. The work of McLaughlin et al. involves significant extensions of the steepest descent method: Such extensions are needed in order to overcome the new difficulties introduced into the problem by the singularity of the potential $V(x)$ at $x = 0$.

Finally, there are problems in which the asymptotic behavior of the system at hand is described by OP's. This happens, in particular, in the case of the so-called Toda rarefaction problem (see [DKKZ]). Here one considers the initial-boundary value problem for the Toda lattice

$$\ddot{x}_n = e^{x_{n-1}-x_n} - e^{x_n - x_{n+1}}, \quad n \geq 1 \tag{120}$$

where for some $\alpha > 0$

$$\begin{cases} x_n(0) = \alpha n, & n \geq 1; \\ \dot{x}_n(0) = 0, & n \geq 1, \end{cases} \tag{121}$$

and the driving particle moves with a fixed velocity $2a$

$$x_0(t) = 2at, \qquad t \geq 0. \tag{122}$$

Making the change of variables $x_n \to \alpha n + y_n$ one sees that, apart from rescaling time, one can always assume without loss of generality that $\alpha = 0$ in (121). One may think of (120)–(122) as a cylinder of particles $\{x_n\}_{n\geq 1}$ driven by a piston x_0. If $a > 0$, one has the (Toda) shock problem ([VDO]) and if $a < 0$ one has the (Toda) rarefaction problem. In the rarefaction problem, if $|a|$ is sufficiently large ($|a| > 1$ turns out to be the critical region) one expects that the piston will separate from the "gas" $\{x_n\}_{n\geq 1}$ and cavitation will occur. This is indeed what happens: if $a < -1$, the authors in [DKKZ] show, using the RH steepest descent method, that as $t \to \infty$, the solution of the Toda lattice splits into two parts, I+II. Part I models the cavitation and Part II is an exponentially decreasing error term. Quite remarkably, Part I is constructed from the solution of an associated OP problem, which turns out to be the Fokas, Its, Kitaev RHP in disguise. We refer the reader to [DKKZ] for details.

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